

Invariants and Mathematical Structuralism[†]

GEORG SCHIEMER*

The paper outlines a novel version of mathematical structuralism related to invariants. The main objective here is twofold: first, to present a formal theory of structures based on the structuralist methodology underlying work with invariants. Second, to show that the resulting framework allows one to model several typical operations in modern mathematical practice: the comparison of invariants in terms of their distinctive power, the bundling of incomparable invariants to increase their collective strength, as well as a heuristic principle related to the search for new invariants.

1. Introduction

Mathematical structuralism is the philosophical position that mathematics studies abstract structures. According to this view, mathematical theories describe the structural properties of their respective domains without regard for intrinsic qualities of the objects in question. Peano arithmetic thus describes the natural-number structure (the \mathbb{N} -structure), the axioms for complete ordered fields describe the real-number structure (the \mathbb{R} -structure), geometry the structure of (Euclidean) space and so on.

There are basically two motivations for such an account of mathematics. The first one is genuinely philosophical. It concerns issues relevant to the epistemology and ontology of mathematics, for instance the true nature of the natural numbers and our ways of referring to them. Theories of structuralism addressing these philosophical issues originated with [Benacerraf, 1965]. The second motivation has roots further back and lies not in philosophy but in mathematics itself. One can understand structuralism as an attempt to come to terms philosophically with a number of wide-ranging methodological transformations in mathematical practice in the nineteenth

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* Munich Center for Mathematical Philosophy (MCMP), Ludwig-Maximilians-Universität, 80539 Munich, Germany. georg.schiemer@univie.ac.at

and twentieth centuries, particularly the rise of non-Euclidean geometries, of abstract algebra, and of formal axiomatics.¹ Thus, independent of the intrinsically philosophical considerations, there are genuinely mathematical reasons to adopt a structuralist view, namely the different ‘structuralist methodologies’ [Reck and Price, 2000] in modern mathematics itself.

Mathematical structuralism today comes in a variety of forms, based on different conceptions of the notion of a ‘mathematical structure’.² Despite the conceptual differences, most present theories in the tradition of Benacerraf’s paper are mainly concerned with the genuinely philosophical issues mentioned above. The main topics discussed in the recent literature reflect this focus of attention: the metaphysical nature of objects in a structure (e.g., [Shapiro, 2000]); the dependence between objects in a structure (e.g., [Linnebo, 2008]); or the identity problem concerning structurally indiscernible objects (e.g., [Leitgeb and Ladyman, 2008], [Shapiro, 2008]).

In turn, significantly less attention has been drawn so far to the structuralism immanent in modern mathematics. This is so in spite of the recent increase of interest in philosophical issues concerning mathematical practice.³ The present paper is an attempt to bridge this gap and to treat structuralism as a contribution to the philosophy of mathematical practice. Specifically, it will outline a new account of structures that describes a particular structural practice present in contemporary mathematics.

The present theory differs from existing work in two respects. The first difference concerns the notion of a mathematical structure. So far, this notion has usually been identified with the concept of *axiomatic* or *axiomatized* structure, viz., with the concept of an abstract object specified by a certain axiom system. The \mathbb{N} - and \mathbb{R} -structures are cases in point here. However, these are certainly not the only kind of structures relevant in modern mathematics. The focus will therefore be on a different, more fine-grained notion than axiomatic structure. Specifically, as will be shown in Section 5, one particular way in which mathematicians understand structural information about objects like groups and graphs is closely connected to work on mathematical invariants.

The second difference concerns the focus on relations *between* mathematical structures, that is, on *inter-structural* relations. In the present

¹ See [Corry, 2004] for an in-depth study of the development of abstract algebra. Compare [Reck and Price, 2000] as well as [Maddy, 2008] for general discussions of the structuralist turn in mathematics in the period in question.

² See [Reck and Price, 2000] and [Cole, 2010] for overview surveys on current positions and important research questions.

³ Important exceptions are [Mac Lane, 1996; McLarty, 2008; Carter, 2008] and, more generally, recent contributions to category-theoretic structuralism. See below for further discussion and for references. For recent work on the philosophy of mathematical practice, compare [Mancosu, 2008].

literature, there is an almost exclusive focus on the study of specific structures — for instance structures defined by categorical theories like (second-order) Peano arithmetic — and of relations among abstract places *within* them.⁴ In contrast, comparatively little has so far been said about theories describing more than one structure and about possible relations *between* them. For instance, noncategorical theories like group or ring theory are usually not investigated in detail from a structuralist perspective. The present account will explicitly deal with different structures of a given mathematical theory. Moreover, Section 4 will be dedicated to the explication of several inter-structural relations between them.

The main objective of this paper is twofold: first, to outline a structure theory based on mathematical invariants. Second, to show that the resulting framework allows one to model formally or explain several aspects of the structuralist methodology underlying work with invariants: the comparison of invariants in terms of their relative distinctive power, the bundling of incomparable invariants to increase their collective strength, as well as certain issues in connection with the heuristics underlying the search for new invariants.

The present theory of invariant-based structuralism will turn out to be comparable to recent work on *category-theoretic* (or simply *categorical*) structuralism, both in terms of its general motivation as well as with respect to the above two characteristics.⁵ As the name suggests, categorical structuralism comprises different attempts to explicate the structure of particular types of mathematical objects (such as groups, graphs, or sets) in purely category-theoretical terms. Moreover, as will be shown in Section 5, category theory is concerned not only with the specification of the general structures of categories but also with the study of relations *between* them (usually expressed in terms of functors). Finally, in contrast to other approaches, categorical structuralism is less concerned with traditional philosophical issues (as raised in Benacerraf's 1965 paper), but rather aims to capture the 'structural approach' in modern mathematical practice in the language of category theory (see, e.g., [Awodey, 1996]). A third objective in this paper will thus be to explore further these general similarities as well as several differences between the structure theory outlined below and categorical structuralism.

The paper is organized as follows: Section 2 will set the philosophical stage for our theory. We discuss a version of *in re* structuralism, according

⁴ Examples for such *intra-structural* relations are the *dependency* and the *discernibility* between places in a structure. See, e.g., [Linnebo, 2008; Leitgeb and Ladyman, 2008], and [Shapiro, 2008] for details.

⁵ See, e.g., [Awodey, 1996; Hellman, 2003; Awodey, 2004; Rodin, 2011]. See [Cole, 2010] for a recent overview on specific debates within categorical structuralism and for further references.

to which structures are to be formed from more concrete objects in terms of mathematical abstraction. Section 3 will then provide the mathematical background for our theory. We discuss the general form of mathematical invariants as well as simple examples from matrix and graph theory for illustration. Based on this, we attempt to precisify the notion of ‘structural information’ (about particular mathematical objects) frequently used by mathematicians to express the structural content of invariants.

In Section 4, a novel theory of *in re* structures based on this informal invariant structuralism will be outlined. Specifically, we give precise definitions of the notions of ‘property structure’, ‘structural refinement’, and ‘extremal structure’. Several graph-theoretical examples will be presented to illustrate them. In Section 5, this reconstruction of invariant structures will be compared to different versions of categorical structuralism. Finally, in Section 6, it is shown how the new account of *in re* structures can be used to explain the structuralist methodology underlying several typical operations with invariants in mathematical practice. In particular, we discuss here the method of pairing invariants (in order to increase their collective distinctive strength) (6.1) as well as some issues related to the general heuristics of invariant discovery (6.2).

2. *In re* Structuralism and Mathematical Abstraction

In present contributions to mathematical structuralism, little attention is usually paid to noncategorical theories that describe objects of different structures. If mentioned in the literature, philosophers usually remain silent about the very nature of such structures. Instead, it is often held that these theories specify a more general *type* of structure that applies also to non-isomorphic mathematical objects. For instance, group theory is often said to describe *the* group structure (as a higher-order type of structure) that can be instantiated by any concrete group satisfying the group axioms.⁶

There exists an alternative approach that will concern us in the following. The general idea here is to embrace the fact that noncategorical theories define not a single structure but a ‘class of related structures’ of nonisomorphic objects. Two distinct ways to characterize them are presently on the market. One is Shapiro’s well-known *ante rem* structuralism (ARS). *Ante rem* structures are composite abstract objects that can be instantiated by the mathematical models (or ‘systems’) of a theory.⁷

⁶ See, e.g., [Shapiro, 2000, p. 73] for such a view.

⁷ Structures are specified by Shapiro in the following way: ‘A structure is the abstract form of a system, highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system’ [Shapiro, 2000, p. 74]. Different properties of *ante rem* structures are then specified in terms of an axiomatic *structure theory* (analogous to ZF for set theory) [Shapiro, 2000, pp. 92–96].

What is essential is their ontological status as ‘bona fide’ objects, *viz.* the fact that they exist independently of more concrete mathematical objects instantiating them. The second approach denies this ontological independence of structures from more concrete objects. It is usually labeled *in re* structuralism (IRS) and will be the focus in this paper.⁸

According to the particular understanding of IRS discussed here, structures are not conceived of as ‘ontologically prior’ to their instantiating models. They are abstract entities whose existence is strongly dependent on more concrete mathematical objects (such as specific groups, rings, graphs, *etc.*).⁹

This relation of dependency between objects and their structure is generally characterized in terms of some form of *Aristotelian* abstraction, that is, the method of forming new (abstract) objects by removing the nonessential properties of existing (concrete) objects.

More precisely, in the version of IRS investigated here, the concrete objects are usually *systems* of a mathematical theory (in the standard model-theoretic sense), for instance specific groups $\langle G, \circ \rangle$ or the natural-number system $\langle N, 0, <, +, \times \rangle$. *In re* structures are then defined as isomorphism classes (or isomorphism types) of such objects. The relevant notion of mathematical abstraction used to specify them is sometimes called *Dedekind* or *Russellian abstraction*: the method of constructing new abstract properties by taking equivalence classes from a given class of mathematical objects.¹⁰

To illustrate this approach, consider a simple example from abstract algebra. A monoid is a triple $\langle G, \circ, e \rangle$, where G is a set, $e \in G$, and \circ is a binary operation $G \times G \rightarrow G$ that satisfies the following two axioms:¹¹

$$\forall x, y, z \in G : (x \circ y) \circ z = x \circ (y \circ z) \quad (1)$$

$$\forall x \in G : e \circ x = x \circ e = x. \quad (2)$$

⁸ See, *e.g.*, [Shapiro, 2000; Psillos, 2006, p. 91–92; Linnebo, 2008]. An early version of *in re* structuralism very similar to the present accounts was formulated by Rudolf Carnap in the 1920s, in particular in [Carnap, 2000] and [Carnap and Bachmann, 1936]. See [Schiemer, 2012] for further historical details.

⁹ Thus, the kind of IRS discussed here is a noneliminative theory of structures just like Shapiro’s ARS. In contrast to eliminative versions of IRS (such as Hellman’s modal structuralism), *in re* structuralists in the sense specified below are not nominalists, but realists about abstract mathematical structures.

¹⁰ Historically speaking, these labels might be somewhat misleading. The above notion of structure abstraction was introduced in [Russell, 1903] and then described more generally in [Russell, 1919]. It is debatable, however, whether it also corresponds to Dedekind’s original understanding of abstraction. See [Reck, 2003] for a detailed survey and interpretation of Dedekind’s structuralism. Compare [Parsons, 1990] and [Linnebo, 2008] for a discussion of ‘Dedekind abstraction’ in the context of the modern debate.

¹¹ Axiom 1 states that the operation \circ is associative. Axiom 2 states that e is the identity element in G under \circ .

Simple examples of monoids are $\langle N, +, 0 \rangle$ and $\langle R, \times, 1 \rangle$. According to IRS, the abstract structure of a particular monoid can be understood as its isomorphism class, *i.e.*, as the class of monoids isomorphic to it.¹² Structures in this sense are usually specified indirectly in terms of *abstraction principles*. Given two monoids G and H , the relevant principle of structure abstraction will have the following form:

$$[G] = [H] \Leftrightarrow G \cong H. \tag{SA}$$

Informally, this states that G and H share the same structure iff they are isomorphic (that is, if and only if there exists a monoid isomorphism between them). Their abstract structure is thus given or ‘represented’ by the same isomorphism class. The equivalence class notation ‘[...]’ can be viewed as an abstraction operator here that assigns a structure to a more concrete mathematical object.¹³ According to IRS, the algebraic theory (*viz.*, axioms 1 and 2) then simply describes the class of different isomorphism classes of the underlying domain of monoids.¹⁴

In a similar set-theoretic version of IRS, sometimes labeled *set-theoretic structuralism*, the cumulative hierarchy of sets V is presupposed as a general background ontology. Thus, a theory like ZFC is assumed as a nonstructural, interpreted background theory in which both mathematical objects and their structures are to be described. Mathematical objects like monoids are represented model-theoretically as structured sets living in V . The structure of a particular monoid G is then specified as the isomorphism type of this set-theoretic representation, that is, as ‘an order-type of [its] model-theoretic interpretation’ [Shapiro, 2000, p. 272].¹⁵

The general mathematical idea underlying both accounts is this: let \mathcal{K} be a class of mathematical objects defined by a theory \mathbb{T} , that is $\mathcal{K} = \{\mathfrak{M} \mid \mathfrak{M} \models \mathbb{T}\}$. Let \mathcal{K}/\cong be the partition of \mathcal{K} induced by the relevant isomorphism relation \cong in question. For an object $\mathfrak{M} \in \mathcal{K}$, the relevant

¹² We say that two monoids G and H are isomorphic, in symbols $G \cong H$, if and only if there exists a bijective function $f : G \rightarrow H$ such that (i) $\forall a, b \in G : f(a \circ^G b) = f(a) \circ^H f(b)$ and (ii) $f(e^G) = e^H$.

¹³ Compare Linnebo on this: ‘(...) the isomorphism types of relations can be used to represent what I called (...) Dedekind abstraction, that is, the operation that maps a system to its abstract structure’ [Linnebo, 2008, p. 75].

¹⁴ In category theory, a different way to approach this is in terms of the duality between the groupoid of models and isomorphisms and the theory of monoids. See [Forszell, 2008] for a study of first-order logical duality.

¹⁵ Shapiro correctly emphasizes that these isomorphism types of mathematical systems are strictly speaking not sets living in V , but proper classes [Shapiro, 2000, p. 92]. Thus, according to set-theoretic structuralism, monoids have set theoretic representations in V , while monoid structures have set-theoretic counterparts of more complex nature, namely as proper classes.

in re structure is *identified* with its isomorphism class $[\mathfrak{M}]$. Each structure of \mathbb{T} is thus a cell $[\mathfrak{M}] \in \mathcal{K}/\cong$. Since \mathcal{K}/\cong forms a partition, any two structures are mutually exclusive. Moreover, the union of structures is collectively exhaustive, that is, (i) $[\mathfrak{M}] \cap [\mathfrak{N}] = \emptyset$, and (ii) $\cup_{\mathfrak{M} \in \mathcal{K}} [\mathfrak{M}] = \mathcal{K}$.

In re structuralism — in both its set-theoretic and its informal version — has two interesting features that distinguishes it from other accounts, in particular from *ante rem* and *categorical* structuralism. First, it is a ‘bottom-up’ theory: *in re* structures are *built* from their instantiating mathematical objects and not *postulated* in a separate structure theory. Specifically, a scheme like (SA) indicates how structures are to be formed by abstracting from the nonstructural features of concrete mathematical objects. IRS thus provides an explicit account of how mathematical structures are related to more concrete mathematical entities, for instance monoids like $\langle N, +, 0 \rangle$. In this sense, it is clearly more constructive in spirit than so-called ‘algebraic’ or ‘top-down’ theories where structures are specified axiomatically without prior regard for the possible systems instantiating them.

Categorical structuralism is clearly of the latter kind. Here, a specific mathematical structure is characterized purely category-theoretically, that is, in terms of morphisms between the objects of the given category. No prior set- or model-theoretic representation of these objects needs to be given. In fact, questions concerning their specific nature are usually taken to be redundant from a category-theoretic perspective. As Awodey puts it, the objects are simply left ‘undetermined’ in the category-theoretic approach. Thus, in terms of the above example, the idea is to specify the ‘monoid structure’ by investigating the category of monoids and monoid homomorphism without considering the set-theoretic construction of particular monoids.¹⁶

In fact, in the category-theoretic study of the semantics of mathematical theories, there seems to be no need to focus exclusively on one category (such as the category of monoids) and thus to treat the systems as prior to their theories. Models can also be understood more generally as functors from a given category encoding a theory to another category, for instance

¹⁶ Compare Awodey on this methodological difference between the ‘top-down description’ of mathematical structures and a more foundational approach based on a ‘bottom-up hierarchy of things’:

This lack of specificity or determination [of particular objects] is not an accidental feature of mathematics . . . Rather it is characteristic of mathematical statements that the particular nature of the entities involved plays no role, but rather their relations, operations, *etc.* — the ‘structures’ that they bear — are related, connected, and described in the statements and proofs of theorems. [Awodey, 2004, p. 59].

the category of sets. In this way, theories (like the theory of monoids) are treated as given first and their models are then specified in terms of functors to other categories. Note that the specification of the ‘structure’ of monoids as well as of isomorphism classes of models then depends crucially on the selection of the particular target category for the models in question. Nonetheless, the description of the objects in the target category will again proceed in a top-down manner, as described above.¹⁷

Shapiro’s theory of *ante rem* structures can also be viewed as a top-down algebraic approach. The term structure is treated as a primitive that is implicitly characterized by several axioms. Specifically, Shapiro’s theory describes a universe of abstract structures with a set of abstract structure-places as *Urelements* [Shapiro, 2000, pp. 92–97]. As mentioned above, the important point is that *ante rem* structures are conceived here as ‘*bona fide*’ objects, that is, as objects that exist independently of their concrete mathematical instantiations. In IRS there is no such independence from the mathematical objects. On the contrary, the specific form of a (set-theoretic) representation of an object directly determines the form of its *in re* structure.

A second difference from ARS is that *in re* structures are conceived as genuinely mathematical objects, namely as isomorphism classes and not as objects of an altogether different ontological kind such as Platonic universals. In this sense, IRS also seems to be more in line with the actual mathematics than ARS. The relation to categorical structuralism in this respect is more difficult to assess. It has been argued that category theory allows a better and more uniform description of structural mathematics than the (informal) set theory underlying IRS.¹⁸ Moreover, categorical structuralism is sometimes viewed as the more consistent philosophical position given that it is a ‘structuralism all the way down’ that does not rest on a non-structural background theory like ZFC (see in particular [Landry, 2011]).

We will not pursue these philosophical issues concerning the proper conceptual basis of modern ‘structural’ mathematics here any further but instead focus on a different aspect of *in re* structuralism and its relation to mathematical practice.¹⁹ The relevant point is that the different versions of IRS presented so far are all limited in one important respect. This is the fact that the underlying notion of a structure — specified in terms of abstraction from isomorphisms — is still too general to model much of the informal talk about structures present in actual mathematical practice.

¹⁷ I would like to thank one of the two anonymous referees for very insightful comments on this issue.

¹⁸ See in particular [Awodey, 2004]. Compare [Rodin, 2011] for an alternative view.

¹⁹ A closer comparison between our version of IRS and categorical structuralism is given in Section 5.

In a certain sense, the conception of structures as isomorphism classes closely reflects the underlying structuralist spirit in mathematical practice given that mathematicians usually do consider objects *up to isomorphism*. In fact, to *identify* isomorphic objects can be viewed as a basic premise of structuralism. However, this principle does not apply universally in modern mathematics. First of all, other more coarse-grained equivalence relations are of equal importance for the comparison of mathematical objects. Homotopy equivalence in algebraic topology is a case in point here. More generally, category-theorists do hold that ‘real’ mathematicians also treat categories, however not up to isomorphism, but rather up to equivalence (see [Awodey, 2010, §7.8]).

Second, the resulting notion of structures *as* isomorphism classes is simply too uninformative in certain contexts to be mathematically relevant. In particular, it usually contains no information about objects that allows one to specify the conditions under which two particular mathematical objects are structurally identical, that is, isomorphic. In such cases, other types of ‘structural information’ about objects are needed that are essentially weaker and less general than the information provided by looking at isomorphism classes.²⁰

Attention to such more coarse-grained mathematical structures is particularly relevant in the context of ‘algebraic’ fields (in Shapiro’s broad sense of the term). For example, groups can share very different structural properties beside the general group structure described by the standard group axioms: they can be *finite*, *infinite*, *simple*, *abelian*, *non-abelian*, *cyclic*, *symmetric*, etc. These structural properties are obviously of central mathematical importance for the classification of groups, but cannot be captured by the *in re* structuralism discussed above. The question therefore arises: what precisely are these more fine-grained notions of structure in use in mathematics? In other words, what do mathematicians *mean* (other than isomorphism classes) when they speak of the structural content of mathematical objects?

3. Structural Information and Invariants

Informal talk about structures is ubiquitous in mathematics. Moreover, as noted in the Introduction, it is certainly not limited to structures as defined by axiomatic theories (such as the \mathbb{N} - or \mathbb{R} -structure). There are various different senses in which the notion is understood here.²¹ For instance,

²⁰ A related point has recently been emphasized in [Rodin, 2011] in the context of categorical structuralism. See, in particular, his discussion of invariance under isomorphic transformation and the role of isomorphism in structural mathematics in his §5.

²¹ The fact that different uses of the term ‘structure’ can be found in mathematics has first been stressed in [Mac Lane, 1996] and, more recently, in [Carter, 2008].

mathematicians often speak of (a piece of) ‘structural information’ about a particular object, for instance a specific finite group or graph. The notion of structure intended in these cases is usually associated with mathematical invariants.²²

Generally speaking, given a space or category of mathematical objects, an *invariant* is a function from this space to another set, for instance a set of numbers like N (or R), such that two objects in the domain of the function have the same value if they are structurally similar or equivalent. Structural similarity is usually characterized in terms of an isomorphism relation whose precise definition depends on the particular mathematical category in question. More formally, given a class of mathematical objects D , an invariant is a function $f : D \rightarrow N$ such that for any two objects $X, Y \in D$, we have:

$$X \cong Y \Rightarrow f(X) = f(Y). \tag{Inv}$$

Note that the elements in D are usually mathematical objects like groups or graphs that already possess some kind of internal structure. The symbol \cong expresses the relevant isomorphism relation for the category in question here. For instance, if D is the class of finite graphs, then an invariant function specifies a graph property such that any two graphs will be assigned the same value, *e.g.*, the same natural number, if there exists a graph isomorphism between them.

Furthermore, an invariant or a set of invariants is usually called *complete* if it discriminates between any two non-isomorphic elements in the category in question. In this case, also the right-to-left conditional, from invariance to isomorphism, holds: two objects receive the same output by an invariant function (or by every invariant in the set considered) *if and only if* they are isomorphic. More formally, a set of invariants $\{f_i : D \rightarrow N_i\}_{i \in I}$ (with index set I) is complete iff for any two objects $X, Y \in D$ we have:

$$X \cong Y \Leftrightarrow f_i(X) = f_i(Y), \text{ for every } i \in I. \tag{CInv}$$

Complete and incomplete invariants of this form play a central role in contemporary mathematics. For instance, the search for new invariants or for a complete set of invariants form lively research fields in graph theory, topology, and knot theory. Moreover, invariants have several important mathematical applications, for instance in structure theorems (*e.g.*, for

²² This connection between invariants and ‘structures over sets’ of mathematical objects has first been pointed out in [Carter, 2008]. See Section 4 for further discussion of Carter’s invariant structuralism and of how it relates to the account presented in this paper.

nonabelian groups) or in classification theorems (such as in the famous classification theorem for finite simple groups).²³

To illustrate their general logical form, we discuss two simple examples of invariants from different mathematical fields. The first one is from linear algebra:

Example 1. Trace of a matrix

A simple invariant used here is the *trace* of a square matrix. Given an $n \times n$ matrix $C = (c_{ij})_{0 \leq i, j \leq n}$, the trace is defined as the sum of elements c_{ii} of its diagonal:

$$\text{Tr}(C) = \sum_{i=1}^n c_{ii} = c_{11} + c_{22} + \dots + c_{nn}.$$

The trace is a so-called *similarity invariant*. Similarity is an equivalence relation on the space of square matrixes that preserves the structure of a matrix: two matrices C, D are called *similar* if there exists an *invertible* $n \times n$ matrix P such that $C = P^{-1}DP$.²⁴ The trace is thus a function of the form $\text{Tr}: \mathcal{M} \rightarrow N$ that produces the same output for any two similar matrices $C, D \in \mathcal{M}$:

$$C \sim D \Rightarrow \text{Tr}(C) = \text{Tr}(D).$$

The second example comes from graph theory. The broad spectrum of existing invariants here includes very simple graph properties such as the *order* n (the number of vertices) or the *size* q (the number of edges) of a graph. An invariant that is a bit more complex is the *clique number* of a graph:

Example 2. Clique number of a graph

Given a graph $G = \langle V, E \rangle$ with vertex set V and edge set E , a clique is a subset of the vertices $X \subseteq V$ such that $\forall u, v \in X: \{u, v\} \in E$. A maximum clique is a largest clique in G . The clique number ω is the number of vertices of a maximum clique in a graph. It is a function of the form

²³ In the latter, a complete set of invariants has recently been specified that gives an exhaustive classification (up to isomorphism) of all finite simple groups. Compare Aschbacher on the general use of invariants here: ‘The usual procedure for classifying a collection of objects is to associate to each member of the collection some family of invariants, prove that each object is uniquely determined by its invariants, and determine which sets of invariants actually correspond to objects’ [Aschbacher, 2000, p. 260].

²⁴ A related equivalence relation in abstract algebra is the notion of conjugacy in (non-abelian) group theory. Given a group G , we say that two elements $f, g \in G$ are conjugate if there exists an element $h \in G$ such that $hgh^{-1} = f$. Group G can thus be partitioned into its conjugacy classes. A corresponding invariant is then the *class number*, i.e., the number of conjugacy classes of G . See, e.g., [Lang, 2002] for details.

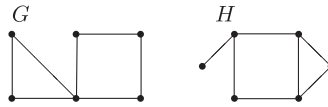


FIG. 1. $\omega(G) = \omega(H) = 3$

$\omega : \mathcal{G} \rightarrow N$ (where \mathcal{G} denotes the space of graphs) such that any two isomorphic graphs G and H share the same clique number:

$$G \simeq H \Rightarrow \omega(G) = \omega(H).$$

Consider, for example, two simple graphs G and H (see Figure 1). It can easily be seen that both graphs share the same clique number, namely $\omega(G) = \omega(H) = 3$. Nevertheless, G and H are clearly nonisomorphic, thus ω does not form a complete invariant for the space of graphs.

Both clique number and trace are so-called *numerical* invariants, that is, invariants valued in specific number sets. In both examples, the codomain of the invariant function is N . This is not necessarily the case. In fact, many of the more complex invariants investigated in mathematics range over more complex objects than the natural or real numbers. One case in point here is *algebraic* invariants such as the *spectrum* of a graph, *i.e.*, the multiset of eigenvalues of a graph’s adjacency matrix. Another case in point is *polynomial* invariants. *Graph* and *knot polynomials* such as the Jones polynomial are functions that assign polynomials to the objects in question. The ranges of these invariants are rings of polynomials like $\mathbb{Z}[X]$ or $\mathbb{R}[X]$ (for some set of indeterminates X).²⁵

An altogether different type of invariants that plays a central role in contemporary mathematics comprises so-called *functorial* invariants. As the name suggests, these invariants are not functions between sets but functors between mathematical categories that are usually specified in terms of a ‘*categorification*’ of existing functional invariants. (This point will be discussed in further detail in Section 5.) Examples of such invariants are (singular) homology or cohomology in algebraic topology which both result from a categorification of the Euler characteristic of topological spaces. In these cases, the respective invariant is a functor from the category of topological spaces to the category of groups that assigns (co-)homology groups to spaces. Another more recent example of a functorial invariant in knot theory is the Khovanov homology that presents a categorification of the Jones polynomial.²⁶

²⁵ See [Hatcher, 2002] for a textbook presentation of topological invariants and [Averbouch, 2011] for a detailed discussion of several graph polynomials.

²⁶ See again [Hatcher, 2002] for a detailed discussion of the functorial topological invariants. For details on Khovanov homology, see [Khovanov, 2000]. Compare also Section 5 below for further references concerning the method of categorification in mathematics.

Finally, invariants can differ from the two examples discussed above in the sense that the equivalence between mathematical objects that is preserved by the invariant is not an isomorphism, but a more coarse-grained equivalence relation. For instance, topological spaces are often not treated up to homeomorphism, but up to homotopy equivalence. Accordingly, homotopy invariants such as the fundamental group or homology describe topological properties that hold (or fail to hold) of all spaces of the same homotopy type. In the case of algebraic invariants like homology, the invariant for spaces X, Y thus has the following form:

$$X \simeq Y \Rightarrow H_n(X) \cong H_n(Y).$$

This simply states that the n th homology groups for X and Y are isomorphic if X and Y are of the same homotopy type.

Mathematical invariants can differ in these various respects from the simple numerical invariants presented above. Nevertheless, several philosophical points can be made about the latter and their relation to a structuralist conception of mathematics that also apply to the more complex cases.²⁷ Note in particular that invariants like the clique number of a graph are often said to express or to reveal information about the abstract structure of particular objects of a given mathematical category. This informal notion of *invariant-based* structures found in mathematical practice differs from present philosophical accounts of structures in two important respects. (These differences will be relevant for our own formal reconstruction in the following section.)

First, the relationship between mathematical objects and invariant structures is that of a *one-to-many* correspondence: the same object of a given mathematical category can possess a number of different structures, each specified *relative* to a given invariant. Thus, in talk about ‘the structural information of a particular object X ’, the structure in question depends essentially on the specific invariant in question. Each invariant *specifies* or *determines* a particular structural property of the mathematical objects in question. Different invariants extract different pieces of structural information about them.

Second, the structural content expressed by invariants seems to be gradual in character. It allows for differences in grade. Moreover, the structures given by different invariants for the same domain often seem to be related in the sense that an ordering between them can be specified. Thus, one can easily find textbook phrases to the effect that an invariant ‘reveals’ more (or less) structural information about the objects of a given mathematical category than another invariant.

²⁷ For this reason, we will restrict our attention to the discussion of simple numerical invariants in the following.

In the following section, we propose an explication of this informal invariant structuralism based on the three observations made here, namely that (i) different invariants determine different structures on a given mathematical domain of objects, that (ii) these structures can come in different grades, and that (iii) they are sometimes comparable in terms of an ordering.

4. From Invariants to Property Structures

The structure theory presented here is a novel version of IRS that formally models the invariant structuralism presented above (and thus satisfies *desiderata* (i)–(iii)). The general method underlying our approach is based on a *generalization* of the kind of structure abstraction outlined in Section 2. As we will show, abstraction principle (SA) can be generalized in a natural way that allows one to formulate structures as corresponding to particular invariants. Specifically, the aim will be to see how equivalence relations (usually more coarse-grained than isomorphism) on a given mathematical domain are determined by invariants. Based on this, a new type of mathematical structure can be specified by abstraction, that is by taking equivalence classes in the usual sense.

Generally speaking, the present account is motivated by the fact that there is a close connection between IRS based on mathematical abstraction and the informal logic underlying work with invariants. The latter can be understood as abstractions from concrete mathematical objects in the sense that they express information about *particular* structural features of the objects in question. As will be shown below, invariance principles of the general form stated above can be taken to correspond to a generalized notion of structure abstraction that is compatible with IRS.

4.1. Property Structures

An important fact about mathematical invariants is that each invariant for a given class of mathematical objects determines an equivalence relation on that class. More formally, let S be a space of objects, say of graphs or matrices, and let f be an invariant for S . We then say that:

DEFINITION 1. (Determination of an equivalence) *An invariant $f : S \rightarrow N$ determines an equivalence relation $R \subseteq S \times S$ iff for all $X, Y \in S$:*

$$f(X) = f(Y) \iff \langle X, Y \rangle \in R \iff [X]_R = [Y]_R.$$

As mentioned above, the *relata* of such a relation R are usually not unstructured objects but already have some kind of internal structure (such as graphs, matrices, and groups do). In fact, mathematical invariants always specify some property of this internal structure of the objects in

question. For instance, in the category of graphs, the invariant *size* specifies the equivalence relation ‘ x has the same number of edges as y ’ between graphs. Similarly, the invariant *class number* determines the equivalence ‘ x has the same number of distinct conjugacy classes as y ’ for the category of groups.

Given this definition, we can consider any cell $[X]_R$ to be a *structural property* of the mathematical objects in S relative to invariant f . In the above examples, such properties would be ‘a graph with n edges’ (for any n) or ‘a group with n distinct conjugacy classes’ (for any n) in the case of group theory. Informally, every invariant assigns a unique representative from codomain N (for instance, a particular natural number) to each structural property $[X]_R$ given by R . (More precisely, it assigns the same representative to any element in S that shares this structural property). Note that *in re* structures in their usual understanding are special instances of such structural properties, namely those determined by a complete invariant. Put differently, structures (conceived as isomorphism classes) are structural properties if the underlying invariant effectively determines an isomorphism relation on the class of mathematical objects considered.

The central substitute for the notion of *in re* structures in the present reconstruction are not these properties, however, but instead a notion of *property structures* (specified also relative to a given invariant). It is defined as follows:²⁸

DEFINITION 2. (Property Structure) *A pair $\langle S, P \rangle$ (or simply set P) is a property structure of S iff:*

- (i) *there exists an invariant $f : S \rightarrow N$ and an equivalence relation $R \subseteq S \times S$ such that f determines R ; and*
- (ii) *P is the partition of S induced by R , i.e. $P = S/R$.*

Based on this, we say that P is a property structure of S relative to some invariant f for S . Expressed in the terminology of equivalences, let EQ_S be the class of possible equivalence relations on S and Π_S be the corresponding class of possible partitions of S . Each invariant f determines a relation $R \in EQ_S$ and thus also a partition $P \in \Pi_S$ of S . We suggest that structures are to be identified with such partitions of a class of mathematical objects. More generally, one can say that each invariant for a given class of mathematical objects determines a property structure on that class. If the invariant is complete, then the corresponding structure partitions S into its different isomorphism classes.

²⁸ Property structures in a similar sense are discussed in [Leitgeb, 2007]. The difference from our account is that property structures are defined more generally there: they include not only partitions but also sets gained by abstraction from similarity relations, i.e., by taking similarity classes. This is essentially Carnap’s method of ‘*quasi-analysis*’ presented in his *Logischer Aufbau*. See [Leitgeb, 2007, p. 183] for further details.

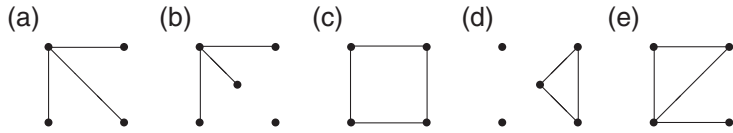


FIG. 2. Set S of finite graphs

This account of property structures has several points of contact with Carter’s discussion of the mathematical structuralism related to invariants in [Carter, 2008]. Here is how she describes the notion of a ‘structure over sets’ used to compute invariants:

The idea behind this is to use the invariant to determine properties of the set. When computing the invariant the aim is to obtain an object that is simpler, but still rich enough to provide the desired information about the underlying set. I shall here emphasize that what is important in this case are the relations that exist between the set, the structure and the invariant. It is because of these relations that it is possible to transfer information from the invariant to the set. [Carter, 2008, p. 124]

The notion of property structures defined above can be viewed as a formal reconstruction of this informal description of the structuralist methodology related to work with invariants. In particular, the information given by an invariant about a class of mathematical objects S that Carter mentions above is essentially the structural information expressed in a property structure. Another point stressed by her is that the information given by a specific invariant does not exhaust the possible structural information about S . This fact is also reflected in our account: different invariants determine different equivalences and thus different property structures on S .

To illustrate these points further, consider the following simple example from graph theory:

Example 3. Graph invariants

Let S be the set of finite graphs with the following elements (see Figure 2). One can compute a number of simple graph invariants for the objects in S . The specific invariants considered here are the *order* n (the number of vertices of a graph), the *size* q (the number of edges), the *independence number* α (the order of a graph’s largest independent set),²⁹ the *clique number* ω (the order of a graph’s maximum clique), the *maximum degree* Δ (the maximum degree of the vertices in a graph), the *minimum degree*

²⁹ An independent set of a graph is a subset of its vertex set such that no two members of it are connected by an edge.

δ (the minimum degree of the vertices in a graph), as well as the *degree sequence* D (the nonincreasing sequence of all degrees in a graph).³⁰

For instance, graphs a, c, e have order 4, graphs b, d have order 5. The clique number of graphs a, b, c is 2, that of d, e is 3. The independence number of a, d is 3, of b is 4, and of graphs c and e is 2. The maximum degree of a, b, e is 3 and of c, d is 2. Finally, the respective degree sequences are: $D(a) = (3, 1, 1, 1)$; $D(b) = (3, 1, 1, 1, 0)$; $D(c) = (2, 2, 2, 2)$; $D(d) = (2, 2, 2, 0, 0)$; $D(e) = (3, 2, 2, 1)$.

By Definition 2, each invariant determines a property structure for set S :

$$\begin{aligned} P_n &= \{\{a, c, e\}, \{b, d\}\}; & P_q &= \{\{a, b, d\}, \{c, e\}\}; \\ P_\omega &= \{\{a, b, c\}, \{d, e\}\}; & P_\alpha &= \{\{a, d\}, \{b\}, \{c, e\}\}; \\ P_\Delta &= \{\{a, b, e\}, \{c, d\}\}; & P_\delta &= \{\{a, e\}, \{b, d\}, \{c\}\}; \\ P_D &= \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\} \end{aligned}$$

Two points of commentary are in order here. First, the above example of invariant-based structures is clearly an idealization of real mathematics in the sense that the invariants discussed here are restricted to a toy set of graphs S . Usually, they are defined for the full space of graphs \mathcal{G} , that is, as functions of the form $f : \mathcal{G} \rightarrow N$. Consequently, the property structures determined by invariants for the full space of graphs are much more comprehensive partition classes than in the above case.

Second, the theory of property structures illustrated here can be viewed as a version of IRS. The reason for this is that structures are directly specified by mathematical abstraction, *i.e.*, by the method of taking equivalence classes. The main difference from existing accounts of IRS is that property structures are not *identified* with these equivalence classes, but instead with set-theoretical entities of a higher rank: they are defined as partitions, that is, as classes of equivalence classes. Moreover, the equivalences in question are usually not isomorphisms, but more coarse-grained relations determined by invariants.

4.2. Structural Refinement and Extremal Structures

The present framework allows one to specify several interesting inter-structural relations between property structures. The mathematical background for their formulation is again the theory of equivalences. As is well known, one can give an ordering of the class of possible equivalence relations EQ_S on a given set S in terms of a *refinement* relation: given two equivalences $R_1, R_2 \in EQ_S$, we say that R_1 is *finer* than

³⁰ See [Diestel, 2005] for more a detailed discussion of these graph invariants.

(or as fine as) R_2 , in symbols $R_1 \preccurlyeq R_2$, iff for any elements $X, Y \in S$: $\langle X, Y \rangle \in R_1 \rightarrow \langle X, Y \rangle \in R_2$.³¹ The ordering given by \preccurlyeq is not total since there might exist *incomparable* equivalences on S .³²

Given this, we can introduce some new structuralist terminology. First, we define an inter-structural relation on a given set of property structures that is directly based on the notion of equivalence refinement.

DEFINITION 3. (Structural refinement) *Let P, Q be two property structures on S induced by the equivalence relations R_1, R_2 (and invariants g, f). Then P is finer than (or equally fine as) Q , in symbols $P \sqsubseteq Q$, iff*

- (i) *for any two objects $X, Y \in S$: $g(X) = g(Y) \Rightarrow f(X) = f(Y)$ (or, equivalently)*
- (i') *for any $X \in S$ and two structural properties $[X]_{R_1} \in P, [X]_{R_2} \in Q$: $[X]_{R_1} \subseteq [X]_{R_2}$.*

Structural refinement between property structures can thus be specified in two ways. It can be defined in terms of a relation between the two corresponding invariants. We say that a structure P is finer than structure Q if and only if invariant g (corresponding to P) has more ‘discriminatory strength’ (or more ‘distinctive power’) than invariant f . Relative invariant strength — as it is understood here — refers to the capacity of an invariant to discriminate between nonisomorphic mathematical objects (of a given class S). Thus, to say that g is stronger than f simply means that g distinguishes between more nonisomorphic objects in S than f . Relative invariant strength is thus equivalent to refinement on the level of property structures: the finer a structure, the stronger its corresponding invariant and vice versa.

Alternatively, structural refinement can also be defined directly in terms of the partitions induced by equivalences R_1 and R_2 : one property structure is finer than another if each equivalence class of the former is a subset of an equivalence class of the latter. Thus, to say that a structure is more refined than another is simply to say that the corresponding invariant determines a finer partition of S than that determined by the other invariant.³³

³¹ A direct consequence of this is the following result: Let R_1, R_2 be two equivalence relations on S . If $R_1 \preccurlyeq R_2$ then for any cell $[X]_{R_2} \in S/R_2$: $[X]_{R_2} = \cup\{[Y]_{R_1} \mid \langle X, Y \rangle \in R_2\}$. For a proof see [Antonelli, 2010, p. 280].

³² To illustrate this, consider the following simple example: consider set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and three equivalence relations R_1, R_2 , and R_3 that induce the following partitions of S : $S/R_1 = \{\{1, 2\}, \{4, 5, 8\}, \{6, 7\}, \{3\}, \{9\}\}$; $S/R_2 = \{\{1, 2, 6, 7\}, \{3\}, \{4, 5, 8, 9\}\}$; $S/R_3 = \{\{2, 5, 7\}, \{1, 6\}, \{3, 4, 8, 9\}\}$. It follows from the above definition that $R_1 \preccurlyeq R_2$ while R_1 and R_3 (as well as R_2 and R_3) are noncomparable.

³³ The equivalence of the two definitions of structural refinement — in terms of relative invariant strength and in terms of equivalence refinement — can easily be shown. Assume first that invariant g is stronger than invariant f . Then for any two objects $X, Y \in S$:

An important mathematical fact about this structuralist framework needs to be stressed here: the binary ‘structural refinement’ relation induces a *partial ordering* on any given set of property structures PS of a given mathematical domain. Thus, for any pair of the form $\langle PS, \sqsubseteq \rangle$, where $\sqsubseteq \subset PS \times PS$ is our refinement relation, the following conditions hold:³⁴

- (i) $\forall R \in PS: R \sqsubseteq R$ (reflexivity) ;
- (ii) $\forall R, S, T \in PS: R \sqsubseteq S \wedge S \sqsubseteq T \Rightarrow R \sqsubseteq T$ (transitivity) ;
- (iii) $\forall S, T \in PS: S \sqsubseteq T \wedge T \sqsubseteq S \Rightarrow S = T$ (antisymmetry).

Note that the partial order given here is not *strict* since two property structures may be of equal strength, *i.e.*, equally structurally fine. It is *partial* (thus not *total*) given that there might be incomparable structures, to be defined in the following way:

DEFINITION 4. (Incomparable structures) *Two property structures P and Q are incomparable, in symbols $P \parallel Q$, if neither $P \sqsubseteq Q$ nor $P \supseteq Q$ holds.*

Based on the notion of structural refinement, we can further specify several mathematically relevant subsets of a given set of property structures. Following an early suggestion made by Carnap, we will dub them extremal structures.³⁵ They are defined in the following way:

DEFINITION 5. (Extremal structures) *Let PS be a set of property structures of the mathematical domain S . We say that*

1. *P is a maximal property structure in PS , in symbols $P \in \text{Max}(PS)$, if there exists no other $Q \in PS$ such that $P \sqsubseteq Q$.*
2. *P is a minimal property structure in PS , in symbols $P \in \text{Min}(PS)$, if there exists no other $Q \in PS$ such that $Q \sqsubseteq P$.*
3. *P is an isolated property structure in PS , in symbols $P \in \text{Iso}(PS)$, if for every other $Q \in PS$ we have $P \parallel Q$.*

The fact that the \sqsubseteq -relation induces a partial ordering on a given set of property structures allows one to reformulate these extremal structures in more standard order-theoretic terminology: the minimal and maximal property structures in a given set PS are simply the *minimal* and *maximal*

$g(X) = g(Y) \Rightarrow f(X) = f(Y)$. By our Definition 3, we have $\langle X, Y \rangle \in R_1 \Rightarrow \langle X, Y \rangle \in R_2$. Thus, R_1 is a subrelation of R_2 . Since both relations are equivalences on S , it follows that R_1 is also a refinement of R_2 and thus $P \sqsubseteq Q$. The other, right-to-left direction is obvious.

³⁴ This is due to the well-known fact that a refinement relation on the class of partitions of a set gives a partial ordering of that class. See, *e.g.*, [Grätzer, 2011, p. 251].

³⁵ An alternative theory of extremal structures in the context of IRS was introduced by Carnap in [Carnap and Bachmann, 1936]. Carnap’s specific understanding of extremal structures differs significantly from the present account and will not be discussed here. See [Schiemer, 2012] for further details.

elements in the poset $\langle PS, \sqsubseteq \rangle$.³⁶ As mentioned above, the relation \sqsubseteq does not induce a total order since there might be ‘isolated structures’, that is pairs in PS that are not *relata* of \sqsubseteq . Put in order-theoretic terms, this simply expresses the fact that there might exist *noncomparable* elements in PS .

To illustrate this account of extremal structures further, consider again our toy example of invariants for a set of finite graphs:

Example 4. Graph invariants, cont’d

Recall from Example 3 that the invariants *order* n , *size* q , *independence number* α , *clique number* ω , *maximum degree* Δ , *minimum degree* δ , and *degree sequence* D determine the following property structures on set S of finite graphs:

$$\begin{aligned} P_n &= \{\{a, c, e\}, \{b, d\}\}; & P_q &= \{\{a, b, d\}, \{c, e\}\}; \\ P_\omega &= \{\{a, b, c\}, \{d, e\}\}; & P_\alpha &= \{\{a, d\}, \{b\}, \{c, e\}\}; \\ P_\Delta &= \{\{a, b, e\}, \{c, d\}\}; & P_\delta &= \{\{a, e\}, \{b, d\}, \{c\}\}; \\ & & P_D &= \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\}. \end{aligned}$$

Based on the above definitions, they can be ordered in the following way:

- (i) several structures are noncomparable: $P_n \parallel P_q \parallel P_\omega$ and also $P_\alpha \parallel P_\omega$
- (ii) several structures are refinements of others (relative to S), for instance: $P_\alpha \sqsubseteq P_q$ and also $P_D \sqsubseteq P_\Delta, P_\delta$.
- (iii) P_D is a *minimum* element in $\langle PS, \sqsubseteq \rangle$, that is it is finer than any other structure in PS .
- (iv) $\langle PS, \sqsubseteq \rangle$ thus includes the following extremal property structures: $P_D \in \text{Min}(PS)$ and $P_n, P_q, P_\Delta, P_\omega \in \text{Max}(PS)$.

The structure theory outlined above is intended as an explication of the implicit structuralism present in mathematical work on invariants. Specifically, it is devised to model formally two important operations with invariants that occur frequently in mathematical practice: the comparison of invariants in terms of their relative discriminatory strength on the one hand; and the search for the invariant(s) with the highest distinctive power in a given set of invariants on the other.

We can see from the example that the refinement relation on property structures clearly explicates relative invariant strength. For instance, $P_\alpha \sqsubseteq P_q$ in Example 4 represents the fact that invariant α is stronger than invariant q . Note again that results like this one are limited in the sense

³⁶ Generally, given a poset P , $a \in P$ is called a maximal element, in symbols $a \in \text{Max}P$ if for any $x \in P : a \leq x$ then $a = x$. Dually, $b \in P$ is called a minimal element, in symbols $b \in \text{Min}P$ if for any $x \in P : x \leq b$ then $b = x$ (see, for instance, [Davey and Priestley, 2002, p. 16]).

that they hold only relative to the small set of graphs S . Thus, α is stronger than q *only* relative to S but certainly not in the full space of graphs \mathcal{G} . Nevertheless, cases of structural refinement can also be found in the more general, non-relativized, sense. A case in point here is $P_D \sqsubseteq P_\Delta, P_\delta$ which is not only true for S , but also for \mathcal{G} . The structural refinement relation thus explicates invariant strength also on the general level, in particular for so-called *extended* or *generalized* invariants like the degree sequence. Moreover, the specification of extremal structures in the above sense can be viewed as a reconstruction of the mathematical activity of further classifying the elements in a given set of invariants.³⁷ In the following section, we will briefly compare this approach with an alternative explication of invariant structuralism in category-theoretic terms.

5. A Comparison With Categorical Structuralism

The reconstruction of invariant-based structures given above is essentially set-theoretic in nature. Structural properties are represented as equivalence classes of mathematical objects that are determined by invariants. Property structures are represented as the partitions induced by such equivalences. As mentioned in Section 2, two possible objections against such a set-theoretic structuralism can be found in the present literature. The first one is that it is not a ‘structuralism all the way down’, but rests instead on an interpreted theory like ZFC. The second one is that the set-theoretic foundationalism implicit in it is not compatible *conceptually* with the ‘structural approach’ in contemporary mathematics.

A typical reaction to both worries is to turn to category theory and to categorical structuralism. Category theory is often conceived as a more natural framework for structural mathematics than set theory.³⁸ Moreover, its algebraic or ‘top down’ approach makes it non-foundationalist in the sense that no interpreted background theory is needed to specify further the particular objects it talks about. But what exactly is the ‘uniform notion of structure’ that is discussed in categorical structuralism? How is it related to the specific kind of invariant-based structures described above?

Categorical structuralism shares with IRS the view that what matters are not the particular mathematical objects (or their model-theoretic representations) but rather the general structure that presents their ‘invariant form’ [Awodey, 1996]. The central idea in the former approach is that such structures can be specified exclusively in category-theoretic terms, *i.e.*, in terms of *morphism* between the objects of a given category as well

³⁷ Compare, again, [Averbouch, 2011] on comparable work in the context of graph polynomials.

³⁸ Compare, for instance, [Awodey, 1996] on the close connection between category theory and the structuralist conception of mathematics.

as in terms of *functors* between different categories. Thus, the category of groups — consisting of all groups and all group homomorphisms — can be used to characterize the general group structure independently of its specification through the standard group axioms.³⁹

The structure of a given category is usually determined by the specification of certain *categorical* properties. These are simply properties expressible in the language of category theory, that is in terms of objects, morphisms, functors, and so on.⁴⁰ Since both morphisms and functors are by definition structure-preserving, categorical properties turn out to be *structural* in the sense that isomorphic objects (of a given category) always share the same properties. More formally, for any categorical property P and two objects A, B (of category \mathbb{C}), the following holds:⁴¹

$$P(A) \wedge A \cong B \Rightarrow P(B).$$

Intuitively, one can say that the general structure of \mathbb{C} is completely determined if all (or sufficiently many) of these properties have been specified for it.

It should be clear at this point that this practice of characterizing structures category-theoretically is closely related to the specific version of IRS presented above. In fact, the two can be viewed as different accounts of the same mathematical phenomenon, namely the invariant structuralism outlined in Section 3. To see the connection, note that on the one hand, categorical properties are functionally very similar to the set-theoretic properties determined by invariants. In fact, any property expressible in the language of category theory is by definition a structural property — that is one respecting isomorphism — and thus corresponds to a particular invariant for the category in question.⁴²

³⁹ Compare Awodey on the general method underlying this: ‘A category provides a way of characterizing and describing mathematical structure of a given kind, namely in terms of preservation thereof by mappings between mathematical objects bearing the structure in question’ [Awodey, 1996, p. 212].

⁴⁰ See [Awodey, 2010, Ch. 2] for a discussion of several examples of categorical properties.

⁴¹ Compare again Awodey on this understanding of structural properties: ‘Now, two objects should be said to bear the same structure just if they are structurally indistinguishable, *i.e.*, just if any ‘structural property’ enjoyed by one is also enjoyed by the other. This is ensured simply by taking the vague, intuitive notion of a structural property to be one that respects isomorphism in the category at issue’ [Awodey, 1996, p. 214]. The very same notion of structural properties can already be found in Carnap’s early work on general axiomatics in [Carnap, 2000].

⁴² Awodey explicitly mentions the relation between categorical properties and invariants at one point: ‘Thus, for example, a structural property of spaces is just one that respects homeomorphism, more usually called a topological property or invariant, such as having a particular fundamental group’ [Awodey, 1996, p. 214]. The close connection between

On the other hand, many of the invariants frequently used in mathematics can be given a category-theoretic representation. In particular, it can be shown that invariants can be explicated in terms of categorical properties. We saw above that an invariant is usually understood as a function from one mathematical domain (of groups, graphs, knots, *etc.*) to another (of numbers, matrices, polynomials, *etc.*) that assigns the same value to isomorphic objects. A natural way to represent invariants in this sense in the language of category theory is by the use of *functors*, *i.e.*, by structure-preserving mappings *between* different categories. Intuitively speaking, the idea is to define a functor (or, in the case of more complex invariants, a number of functors) of the form

$$F : \mathbb{C} \rightarrow \mathbb{D}$$

that maps objects of a given category to objects (of another category) that codify some piece of structural information about them. Thus, \mathbb{C} contains the mathematical objects the invariant is defined for. Their ‘pictures’ in \mathbb{D} are usually gained from the objects of \mathbb{C} by some form of *structure extraction*. Since functors preserve category structure, a structural identity in the domain of the functor is then translated into another type of structural identity in the codomain. More specifically, if there exists an isomorphism between two objects $A, B \in \mathbb{C}$, the corresponding objects $F(A), F(B) \in \mathbb{D}$ are also *unique* up to isomorphism.

In order to illustrate this approach for the case of numerical invariants, recall that, in graph theory, the *order* n of a graph refers to the number of its vertices.⁴³ The invariant is thus a function of the form: $n : \mathbf{Graph} \rightarrow \mathbb{N}$ such that, for any two graphs G, H , $n(G) = n(H)$ if $G \simeq H$. A natural explication of this graph property in category theory is in terms of the composition of two functors that (in combination) codify a kind of structure extraction in the sense specified above. The specific composite functor to represent the *order* of a graph has the following form:

$$F' \circ F : \mathbf{Graph} \rightarrow \mathbf{Set} \rightarrow \mathbf{Card}$$

such that, for any $G \in \mathbf{Graph}$:

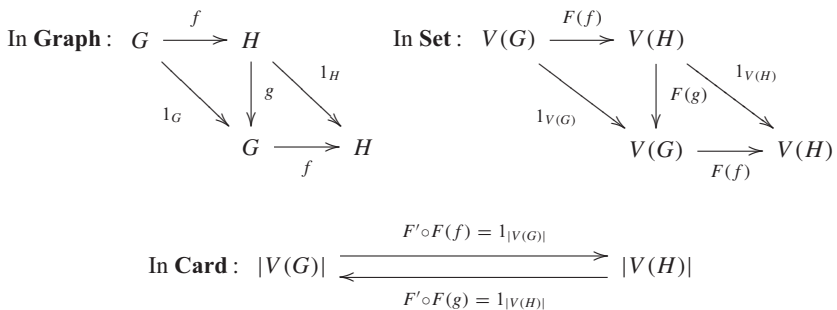
$$F' \circ F : G \mapsto V(G) \mapsto |V(G)|.$$

structural properties and invariants was already observed by Carnap: ‘The structural properties are in a certain sense the invariants under isomorphic transformation. They are of considerable importance in axiomatics’ [Carnap, 2000, p. 74].

⁴³ It should be noted here that there is no uniform category-theoretic representation of invariants simply due to the fact that invariants can be defined very differently. Nevertheless, at least for simple numerical invariants, the reconstruction given below can be taken to be canonical.

Informally, the functor $F' \circ F$ can be taken to extract the structural information ‘the number of vertices’ from every object in **Graph**. Note that functor F maps graphs to their vertex sets. Functor F' in turn maps sets to objects of a specific subcategory, namely the *skeleton* of **Set**. The skeleton of a given category is standardly defined as a subcategory that is (i) full, (ii) isomorphism-dense, and (iii) in which all isomorphic objects are identical.⁴⁴ In the present case, the skeleton of **Set** is the subcategory of all cardinal numbers of sets. The relevant functor F' thus assigns to each set (and, in particular, to each vertex set of a given graph in **Graph**) a specific cardinal number. It is important to stress again that in skeletons, isomorphisms between objects are reduced to identity. Thus, any two objects in **Card** are identical if there exists an isomorphism between them.

The central property of a mathematical invariant, namely that it assigns the same value to isomorphic objects, is captured here by the fact that functors (as well as their compositions) export structural identity. In the present example, this implies that if there is an isomorphism between two graphs, then this fact is preserved by F and F' to hold also for their respective ‘pictures’ in the two other categories. Specifically, it follows that the corresponding vertex sets are isomorphic in **Set**. It follows further that, by the definition of a skeleton, the two cardinal numbers of these sets are identical in **Card**. This fact can be illustrated in terms of commutative diagrams in the following way:



Given this category-theoretic representation of (numerical) invariants, two points of commentary should be added here. First, the fact that invariants like the *order* of a graph can be expressed in terms of functors in the above way suggests that different relations between invariants (such as the notion of relative invariant strength) can also be described category-theoretically,

⁴⁴ A subcategory is said to be isomorphism-dense if every object in the original category is isomorphic to an object in the subcategory. See [Adámek *et al.*, 2004, p. 51] for more detailed definitions of the relevant concepts.

namely in terms of different operations on such functors. This idea will not be pursued here any further.⁴⁵

Second, notice that this way of expressing invariants functorially can be understood as a kind of *decategorification*, that is, a process of reducing categories to sets. This reduction of a category to its skeleton seems closely connected to the kind of structure abstraction described in Section 3. The two functors F and F' provide structural information about graphs by abstracting from all other possible properties of graphs except their order. Thus, the use of category theory in this sense squares well with the more general philosophical account of *in re* structuralism outlined in Section 2.

One should be quick to add at this point, however, that decategorification is certainly not the only way in which category theory is applied to modern structural mathematics. In fact, the reverse process, namely *categorification*, is arguably of much greater importance in contemporary mathematical research. This can be described as ‘the process of finding category-theoretic analogs of set-theoretic concepts by replacing sets with categories, functions with functors, and equations between functions by natural isomorphisms between functors’ [Baez and Dolan, 1998, p. 1].⁴⁶ Thus, whereas decategorification aims at extracting structural information about mathematical objects by reducing categories to sets, categorification can be viewed as the reverse process of adding structural information about particular objects by representing them in a (higher-order) category.⁴⁷

What is interesting in the present context is that categorification also plays an important role in recent research on invariants. The categorification of existing invariants in algebraic topology and knot theory consists, roughly, in the translation of *functional* invariants (valued in sets like N) into *functorial* invariants (valued in categories such as **Group**). The main motivation for constructing categorified versions of existing functional invariants is that they are stronger than the former and thus provide more information about the objects in question. As was already noted in Section 3, homology and cohomology can be viewed as categorifications of the Euler characteristic of topological spaces. In the context of knot theory, Khovanov homology is often mentioned as an example

⁴⁵ Another, somewhat related point concerns the special class of *complete* invariants. A natural candidate to explicate them in category theory would be in terms of so-called *universal mapping properties* (see [Awodey, 1996, p. 221]). In the context of categorical structuralism, one can say that the existence of an universal property allows a complete specification of the general structure of the objects of a given category.

⁴⁶ I would like to thank one of the anonymous referees for bringing this point to my attention.

⁴⁷ See again [Baez and Dolan, 1998] for a detailed discussion of the method of categorification as well as for several mathematical examples.

of a functorial invariant resulting from a categorification of the Jones polynomial.⁴⁸

From a philosophical point of view, it would be interesting to see more closely how this approach of categorification at work in mathematical practice squares with mathematical structuralism. Note that unlike in more traditional accounts of *in re* structuralism, structure abstraction plays no role here. In fact, to ‘categorify’ mathematical objects or concepts (such as invariants) is precisely to reverse this process: it is to enrich their structure by representing them in a higher-order category. Moreover, another central principle of structuralism mentioned above, namely that isomorphic objects are treated as identical, seems to be deliberately weakened here.⁴⁹

We will not discuss these general philosophical issues here any further, but leave them for another paper. Instead we return to the set-theoretical reconstruction of invariant structuralism outlined above. The question addressed in the next section is: what other aspects in modern mathematical research on invariants can be modeled in terms of our structure theory?

6. ... And Back to Invariants

The aim in this section is to show which informal operations with invariants — other than the comparison of individual invariants in terms of their relative strength — can be modeled in terms of our theory of property structures. In particular, the focus here will be on two practices that are typical in work with invariants and can be identified across the mathematical disciplines:

- (1) the study of combinations of invariants;
- (2) the search for new invariants.

Both activities are usually present in research related to the search for a complete set of invariants for a given field. Recall that a (set of) invariant(s) is complete if it allows discrimination between any two non-isomorphic objects of a mathematical domain. In the case of the theory of finite simple groups, a complete set of invariants has been identified in

⁴⁸ See [Hatcher, 2002] for further examples of functorial invariants in algebraic topology. For details on Khovanov homology, see [Khovanov, 2000].

⁴⁹ Compare Baez and Dolan on this point:

One philosophical reason for categorification is that it refines our concept of ‘sameness’ by allowing us to distinguish between isomorphism and equality. In a set, two elements are either the same or different. In a category, two objects can be ‘the same in a way’ while still being different. In other words, they can be isomorphic but not equal. [Baez and Dolan, 1998, p. 7]

recent years.⁵⁰ In other mathematical areas such as graph theory and knot theory, such a set has not been found yet. (But see again the above discussion of the categorification of knot polynomials such as the Jones polynomial in terms of Khovanov homology.) Consequently, there are lively research programs in these fields that concern the search for a completion of the presently known invariants. Work here often includes the study of the distinctive power of collections of existing invariants on the one hand and the search for new invariants that would give such a completion on the other hand.

In the following, we attempt to explain the structuralist methods underlying these two mathematical practices in terms of our structure theory. The basis for this will be a new operation on property structures that corresponds to the act of *bundling* incomparable invariants. Note that in (1) (but also often in (2)), the invariants considered are necessarily incomparable in the sense specified in Section 4. The study of combinations of them in terms of their collective discriminatory strength only makes sense if none of the invariants in question is already known to be weaker or stronger than the other ones in the set. Thus, what mathematicians essentially do in (1) is to compare sets of incomparable invariants with other (sets of) invariants. Translated into our formal framework, they investigate the relation between different sets of property structures that cannot be ordered in terms of a structural refinement relation. Consequently, the main inter-structural relation defined so far does not help to illustrate this comparison of different collections of invariants. Our formal tool-box of inter-structural relations has to be extended.

6.1. Combining Invariants

In order to study combinations of incomparable invariants in the present framework, we can make further use of the order-theoretic structure of a given set of property structures. Recall that any collection of property structures PS (determined by a number of invariants on set S) together with the refinement relation \sqsubseteq forms a partially ordered set. We can graphically represent the precise structure of such a pair $\langle PS, \sqsubseteq \rangle$ in terms of a *Hasse* diagram. The logic of these graphs is straightforward:⁵¹ Each element in PS is represented by a vertex. The vertices representing two elements are connected by a line if one element is *covered* by the other. Given two elements $X, Y \in PS$, we say that X is covered by Y if (i) $X \sqsubseteq Y$ and if (ii) there is no $Z \in PS$ such that $X \sqsubseteq Z \sqsubseteq Y$. Finally, if X is covered by Y , then the point representing X is lower in the diagram than the one representing

⁵⁰ See [Aschbacher, 2000] for further details.

⁵¹ See [Davey and Priestley, 2002, pp. 10–14] for a detailed introduction to Hasse diagrams.

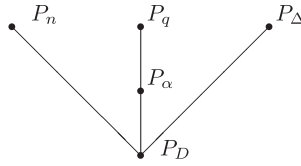


FIG. 3. Hasse diagram of $\langle PS^*, \subseteq \rangle$.

Y (see [Davey and Priestley, 2002, p. 11]). Here is a simple example to illustrate the use of *Hasse* diagrams for our purposes:

Example 5. Hasse diagrams

Consider again Examples 3 and 4 of graph invariants for set S . Let PS^* consist of the property structures $\{P_n, P_q, P_\alpha, P_\Delta, P_D\}$ determined by the invariants $\{n, q, \alpha, \Delta, D\}$. Recall from above that, relative to S , we have $P_D \subseteq P_\alpha \subseteq P_q, P_D \subseteq P_n, P_\Delta$, and $P_q \parallel P_\Delta \parallel P_n$. The corresponding diagram for $\langle PS^*, \subseteq \rangle$ is Figure 3.

The relevant point about the use of Hasse diagrams in the present context is that they allow one to ‘read off’ various relations between property structures easily and thus, indirectly, also between the corresponding invariants. In the above case, the diagram shows that there exists a *minimum* (or *bottom*) element P_D (such that for every other element $X \in PS^* : P_D \subseteq X$) as well as three *maximal* elements P_n, P_Δ, P_q in PS^* . Thus, invariant D is stronger than invariants n, Δ, α , and q (relative to set of graphs S). Moreover, invariants n, q , and Δ are not comparable in terms of their discriminatory strength (relative to S).

What interests us in the present case is not the comparison of individual invariants, however, but the comparison of different collections of invariants. The question is how incomparable invariants work in combination, that is, what their combined strength is relative to another (set of) invariant(s).⁵² A way to spell this out in the present framework can be based on the following idea: given a specific set of property structures (and an ordering in terms of a refinement relation), one can always construct a more comprehensive order-theoretic structure from the original poset. Thus, any PS (for a given class of objects S) can be extended to the more comprehensive set of structures:

$$PS^\diamond := \{S/R \mid \text{for all } R \in EQ_S\}.$$

Informally, PS^\diamond presents the class of all possible structures, *viz.*, all partitions of set S *irrespective* of whether there exist invariants that determine

⁵² Note that the invariants considered here need not be valued in the same sets, *i.e.*, also combinations of invariants with different codomains are permitted. The only thing relevant in the present framework is the fineness of the equivalence relations on the class of mathematical objects that these invariants determine.

them. The relevant mathematical fact is that the extended poset $\langle PS^\diamond, \sqsubseteq \rangle$ forms a *complete lattice*, more specifically a *partition lattice* of S .⁵³ Moreover, the original poset (determined by a concrete collection of invariants) is a submodel of it.

This additional order-theoretic structure can be used for the reconstruction of invariant combinations in the following way: since the extended poset $\langle PS^\diamond, \sqsubseteq \rangle$ forms a complete lattice, it follows that for every subset $B \subseteq PS^\diamond$, the ‘meets’ $\bigwedge B$ and ‘joins’ $\bigvee B$ are also elements in PS^\diamond .⁵⁴ The important philosophical point is that one can now take the newly ‘created’ meets of elements of a given PS to *represent* combinations of property structures and thus, indirectly, also combinations of invariants. More formally, for any set of property structures $B \subseteq PS$ (where each $P \in B$ is determined by an invariant f of a given family of invariants \mathcal{F}), let $\bigwedge B \in PS^\diamond$ express the new structure corresponding to the combination of invariants in \mathcal{F} .

Meets of property structures in this sense are defined in the following way:

DEFINITION 6. (Meets of property structures) *Let $P = \{A_i \mid i \in I\}$ and $Q = \{B_j \mid j \in J\}$ be two property structures of class S (such that $A_i, B_j \subseteq S$). Then*

$$P \wedge Q := \{A_i \cap B_j \neq \emptyset \mid i \in I \text{ and } j \in J\}.$$

More generally, given a set of property structures $\mathcal{P} = \{P_n \mid n \in N\}$ in PS , one can define their meet in terms of the set-theoretic intersection of the partitions: $\bigwedge \{P_n \mid n \in N\} = \bigcap \{P_n \mid n \in N\}$. This is the set of all non-empty intersections of each cell in a given partition with all cells of the other partitions.

To illustrate how such partition meets can be used to represent combinations of property structures, consider the following example of graph invariants:

Example 6. Combinations of property structures

Consider the four graph invariants n, q, α, Δ for set S and the corresponding property structures $P_n, P_\alpha, P_q,$ and P_Δ . Based on them, the following

⁵³ For a general discussion of complete lattices, see [Davey and Priestley, 2002, pp. 33–64]. For a detailed survey of partition lattices, see [Grätzer, 2011].

⁵⁴ Generally speaking, given a poset $\langle P, \leq \rangle$ and $B \subseteq P$, the join $\bigvee B$ is the *least upper bound* of B . Dually, the meet $\bigwedge B$ corresponds to the *greatest lower bound* of B (see again [Davey and Priestley, 2002, pp. 33–35]).

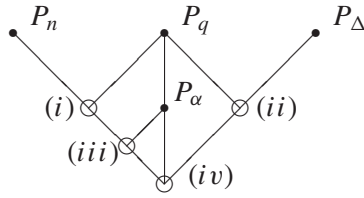


FIG. 4. Meets of property structures $P_n, P_\alpha, P_q,$ and P_Δ .

meets can be formed:

- (i) $P_n \wedge P_q = \{\{a\}, \{b, d\}, \{c, e\}\}$
- (ii) $P_q \wedge P_\Delta = \{\{a, b\}, \{c\}, \{d\}, \{e\}\}$
- (iii) $P_\alpha \wedge P_n = \{\{a\}, \{b\}, \{c, e\}, \{d\}\}$
- (iv) $P_\alpha \wedge P_\Delta = \{\{a\}, \{b\}, \{c\}, \{e\}, \{d\}\}$

The corresponding *Hasse* diagram is Figure 4.

The meet-operation on a set of property structures usually gives us a new structure in an extended poset. This can either be $\langle PS^\circ, \sqsubseteq \rangle$ or a sublattice of it that forms a *completion* of the original $\langle PS, \sqsubseteq \rangle$.⁵⁵ If translated back into the world of invariants, such a refined partition could also be interpreted as corresponding to a new invariant (and thus as a *property structure* in the sense specified in Definition 2). This is actually the case in the above example, where it can be seen that $P_\alpha \wedge P_\Delta = \bigwedge \{P_i \mid i = n, q, \Delta\} = P_D$ for the set of graphs S . However, this is not generally the case. In other words, not every new structure (on a given mathematical domain) introduced by the meet operation actually corresponds to an interesting new invariant for it. It thus seems more adequate — as suggested above — to interpret these newly gained partitions as representing *unions* of existing invariants. Put differently, the method of (downward) completion of a given set of property structures illustrates the process of *bundling* invariants to increase their collective discriminatory strength.

Given this, the comparison of different sets of invariants in terms of their relative distinctive power can also be expressed by relations between the meets of property structures. Specifically, if one set of invariants is collectively stronger than another set, then the single structure corresponding

⁵⁵ In the above example, the extended set of structures is strictly speaking not a completion, *i.e.*, it does not form a lattice as no join operations are considered here. Since we are only interested in cases where the combination of several invariants leads to a structural refinement on the level of property structures in the present context, the focus will be exclusively on the lattice-theoretic meet operation.

to the first set is *finer* than that of the latter (in the formal sense of structural refinement specified above). If the two sets of invariants cannot be related in terms of their discriminatory strength, then the corresponding meets are noncomparable.

In Example 6, it was shown that the combination of the invariants *order* n and *independence number* α has more distinctive power (relative to set S) than the union of invariants *size* q and *order* n , or, more formally: $(P_n \wedge P_\alpha) \sqsubseteq (P_q \wedge P_n)$. In contrast, the respective combinations of invariants $\{n, q\}$, $\{q, \Delta\}$, as well as $\{n, \Delta\}$ are not comparable in terms of their respective strength (for S). On the level of property structures, we have $P_n \wedge P_\Delta = \{\{a, e\}, \{b\}, \{c\}, \{d\}\}$, and, consequently, $(P_n \wedge P_q) \parallel (P_q \wedge P_\Delta) \parallel (P_n \wedge P_\Delta)$.

We give another and more general example, again from the field of graph theory, to illustrate further this way of representing the comparison of sets of invariants:

Example 7. Unions of graph invariants

Consider the graph invariants *order* n , *independence number* α , as well as the *vertex-cover number* τ . A vertex cover of a graph $G = \langle V, E \rangle$ is a set of vertices $X \subseteq V$ such that for any edge $\{u, v\} \in E$: $u \in X$ or $v \in X$. τ expresses the size of a minimum vertex cover of G . It is a well known result about graph invariants that for any graph $G \in \mathcal{G}$, the following holds:⁵⁶

$$n(G) = \tau(G) + \alpha(G).$$

This says that for any given graph in the space of graphs \mathcal{G} , the sum of its minimum vertex cover and its independence number equals its order. It follows from this that for any two graphs $G, H \in \mathcal{G}$, any combination of pairs of invariants in $\{\tau, n, \alpha\}$ has more distinctive power than the remaining single invariant.⁵⁷ More formally,

$$\tau(G) = \tau(H) \ \& \ \alpha(G) = \alpha(H) \Rightarrow n(G) = n(H)$$

$$n(G) = n(H) \ \& \ \alpha(G) = \alpha(H) \Rightarrow \tau(G) = \tau(H)$$

$$\tau(G) = \tau(H) \ \& \ n(G) = n(H) \Rightarrow \alpha(G) = \alpha(H)$$

⁵⁶ This is sometimes called the Gallai Theorem in the graph-theoretic literature (see [Gallai, 1959]).

⁵⁷ For instance, any two graphs of the same order and same independence number also share the same vertex-cover number. Put differently in terms of the underlying set-theoretic operations: the combination of any two invariants can be understood as the intersection of the respective partitions of \mathcal{G} induced by n and α . The above result shows that the cells of this newly created partition are always subsets of the cells of the partition induced by τ .

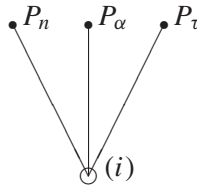


FIG. 5. Structures representing $\alpha + \tau = n$

On the level of the corresponding property structures $P_n, P_\alpha,$ and P_τ (of \mathcal{G}), this result can be expressed by:

$$P_\tau \wedge P_\alpha \sqsubseteq P_n; P_\tau \wedge P_n \sqsubseteq P_\alpha; \text{ and } P_n \wedge P_\alpha \sqsubseteq P_\tau.$$

The relevant *Hasse* diagram has the following simple structure of Figure 5.

This example again shows how the lattice structure of the class of possible property structures (of a given mathematical domain) can be used to illustrate the comparison of different unions of invariants. Note that the original invariants α, τ, n are not comparable with each other. However, all three possible combinations of pairs of invariants are of equal distinctive power. Thus, sets $\{\alpha, n\}, \{\tau, n\},$ and $\{\alpha, \tau\}$ induce the *very* same structure (represented by (i) in the diagram) on \mathcal{G} .⁵⁸ The interesting point here is that this newly gained structure (representing the union of any two invariants) is a refinement of the structure determined by the remaining single invariant in the set.

6.2. Invariant Discovery

In Example 7, each pairing of invariants results in an increase in discriminatory strength relative to the third invariant. However, no such combination gives a complete set that allows one to discriminate between any two non-isomorphic graphs in \mathcal{G} . It is easy to think of graphs that share the same independence number and the same vertex cover (and thus the same order) but that are not isomorphic. In such cases, mathematicians usually search for new invariants that give a completion of the existing set of invariants. Generally, the aim here is to find stronger invariants that allow one to discriminate between a larger number of non-isomorphic objects. At the same time, the newly defined invariant should still be calculable, *i.e.*, it

⁵⁸ It can easily be shown that $P_\tau \wedge P_\alpha = P_n \wedge P_\alpha = P_\tau \wedge P_n$. Let $A \in P_n \wedge P_\alpha$. Thus, for any two $X, Y \in A$ we have $n(X) = n(Y)$ and $\alpha(X) = \alpha(Y)$. By Gallai's result, we also have $X, Y \in A : \tau(X) = \tau(Y)$. Thus $P_n \wedge P_\alpha$ partitions \mathcal{G} into cells of graphs of the same order, independence number, and vertex-cover number. The same partition is effectively given in $P_n \wedge P_\tau$ and $P_\tau \wedge P_\alpha$.

should be decidable whether or not the invariant in question produces the same value for two objects.⁵⁹

What is the underlying methodology of this practice? Is there a general heuristic method of invariant discovery that can be described in terms of our structure theory?⁶⁰ Looking at actual cases of invariant search in mathematics suggests that there is no such uniform method. The heuristics of invariant discovery is a multifaceted and context-dependent mathematical practice: it depends crucially on the nature of the mathematical objects considered. For instance, the search for new invariants in knot theory seems in many ways different from the search for new invariants in group theory, simply because the structural properties of knots — concerning relations between their strands and crossings — are often very different in nature from those of, say, finite simple groups. Consequently, it seems difficult to give a general and unified description of the informal logic underlying the discovery of new invariants. (One such general approach is, as we saw, the categorification of existing invariants, *i.e.*, the construction of new invariants by adding additional categorical structure to them.)

Nevertheless, there is a common heuristic principle that can be identified across the different disciplinary fields and that can be termed the *analysis of failure*. Basically, it is the idea to look at precisely those objects in a given mathematical domain for which an isomorphic mapping fails. Stated informally, the general method is this: one starts with the strongest possible set of given invariants for the field in question and tries to construct a counterexample. Such an example will consist of two objects in the domain, say G and H , that have the same output for each invariant, but are nevertheless nonisomorphic. One then shows why a bijective mapping between G and H fails to be ‘structure preserving’. Based on this diagnosis of failure, one then searches for a new structural property that allows one to discriminate between the two particular objects in question. A new invariant function is formulated that expresses this structural property and gives different outputs for the two objects in question.

This method of constructing counterexamples and analysis of failure can be framed in terms of the present theory of property structures. To see this, consider another simple example from graph theory:

Example 8. Discovery of graph invariants

Let M be the set consisting of the simple graphs of Figure 6.

⁵⁹ See [Gowers, 2008] for a general discussion of the search for new invariants. Gowers holds that these two *desiderata* of invariants — distinctive power and computability — ‘tend to pull in opposite directions’ and that ‘[the] most powerful invariants therefore tend to be the ones that can be calculated, but not very easily [Gowers, 2008, p. 54].’

⁶⁰ The meet operation on property structures specified above does not help here. Lattice-theoretic meets can be used to represent structural refinements given by a combination of existing invariants. They usually say nothing about how new and mathematically interesting invariants are to be discovered.

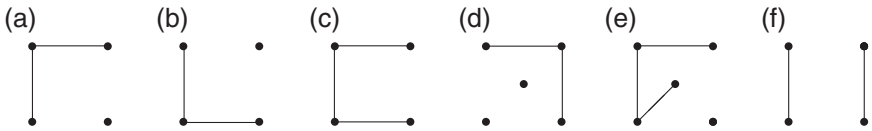


FIG. 6. Set M of finite graphs

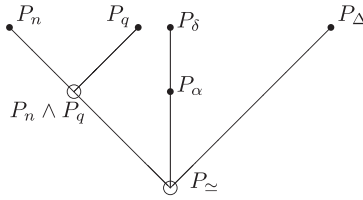
Assume that at time t_0 we only know of the two graph invariants *size* q and *order* n . Their union $\{n, q\}$ determines the property structure $P_n \wedge P_q = \{\{a, b, f\}, \{c\}, \{d\}, \{e\}\}$ on M . It can easily be seen that $\{n, q\}$ does not form a complete set. Specifically, graphs a and b are not isomorphic to graph f , even though they share the same *size* and the same *order*. Thus, the two pairs of graphs $\{a, f\}$ and $\{b, f\}$ can be considered as counterexamples in the above sense.

What distinguishes graphs a, b from f *structurally*? In other words, where does an isomorphism (similar to an isomorphism $a \simeq b$) fail to preserve the structure of f ?⁶¹ The important point to see here is that the adjacency relations of a and b look different from that of f . Specifically, there exists no vertex $x \in V(f)$ that is adjacent to two other vertices $y, z \in V(f)$. This fact suggests several structural properties that discriminate between the graphs. One concerns the *degrees* in a graph, *i.e.*, the number of edges incident to each vertex. One can thus use the two structural properties ‘The highest/lowest degree in a graph’ to discriminate between a, b and f . This is expressed by the two invariants *minimum degree* δ and *maximum degree* Δ discussed above. A third structural property that discriminates between the respective graphs concerns the number of independent vertices: ‘The largest set of independent vertices of a graph’. This corresponds to the invariant *independence number* α .

Based on our analysis of failure of nonisomorphic graphs in M , we have come up with three ‘new’ graph invariants at time t_1 . For the given set of graphs, all three invariants suffice to distinguish between any nonisomorphic elements of M if added to the existing set $\{n, q\}$. Thus, $\{n, q, \alpha\}$, $\{n, q, \Delta\}$, and $\{n, q, \delta\}$ each form a complete set of invariants for M . The respective *Hasse* diagram of the corresponding property structures nicely illustrates this fact (see Figure 7).⁶²

⁶¹ Recall that an isomorphism between two graphs $a \simeq b$ is a bijective function $f : V(a) \rightarrow V(b)$ that preserves the adjacency of the graphs, *i.e.*, for any two vertices $x, y \in V(a) : \{x, y\} \in E(a) \leftrightarrow \{f(x), f(y)\} \in E(b)$.

⁶² Note that $P_{\simeq} = \{\{a, b\}, \{c\}, \{d\}, \{e\}, \{f\}\}$ present the isomorphism structure of M . The property structures determined by the newly introduced invariants are $P_{\alpha} = \{\{a, b, e\}, \{c, f\}, \{d\}\}$, $P_{\Delta} = \{\{a, b, c, d, e\}, \{f\}\}$, and $P_{\delta} = \{\{a, b, d, e\}, \{c, f\}\}$ respectively. The diagram shows that the union of either one of corresponding invariants with the existing set of invariants determines the elements of M up to isomorphism. More formally, $\bigwedge \{P_n, P_q, P_x \mid x = \alpha, \Delta, \delta\} = P_{\simeq}$.

FIG. 7. Completion of $\{q, n\}$ for M

It should be stressed here that this toy example of invariant discovery is limited in several ways: first, it does not give a full description of the heuristics of invariant discovery in mathematical practice. Analysis of failure is certainly not the only method used to discover new invariants. It is one heuristic principle among others that are not discussed here. Second, the search for counterexamples and the analysis of failure undertaken here is fairly easy. In real mathematics, isomorphism testing (and thus the construction of informative counterexamples) is usually a much more difficult task due to the greater structural complexity of the objects considered.

Third, the specific choice of invariants discussed here is rather basic: the ‘newly’ discovered invariants express in fact well known and fairly simple graph properties. In current research in graph or knot theory, new invariants are usually more complex than these simple examples.⁶³ Finally, the example is highly idealized in the sense that each of the newly introduced invariants actually leads to a complete set. In contrast, in actual mathematical contexts it is often not clear whether a new invariant gives such a completion. Proving that a set of invariants is complete — or, in our terminology, that their meet is identical to the isomorphism structure — usually turns out to be more difficult, particularly since the mathematical domain considered is usually far more comprehensive than in our example.

One could infer from this that our theory fails to capture the complexity of actual invariant search in mathematical practice. However, it should be stressed that this is not what is intended in the present context. The intention is *not* to describe the actual heuristic work in the search for new invariants in different mathematical fields but to describe the general ‘structuralist’ method guiding it. Thus, while it is certainly the case that the theory of property structures does not fully explain the search for new invariants, it does nevertheless say something valid about the *implicit* structuralism underlying this kind of research. Put differently, as in the case of studying invariant combinations, our account helps to understand this mathematical activity *as* a structural practice.

⁶³ See, e.g., [Diestel, 2005] for a discussion of more complex graph properties.

7. Conclusion

This paper presented a novel account of mathematical structures. In contrast to several present contributions to mathematical structuralism, the focus here was less on purely philosophical issues concerning the ontology of structures or our epistemological access to them, but rather on the actual understanding of them in mathematical practice. Specifically, we introduced a notion of property structures as well as several ‘inter-structural’ relations between them that make explicit the implicit structuralism underlying work with mathematical invariants.

The philosophical framework for the present account is a specific version of *in re* structuralism, according to which structures are identified with equivalence classes of mathematical objects and specified in terms of abstraction principles. As we saw, property structures can be conceived as *in re* structures in this sense. However, the present approach differs from earlier versions of IRS in that it is based on a *generalization* of the typical abstraction scheme (SA): in our framework, structure abstraction from a given mathematical domain does not have to be based on isomorphism. Any equivalence on that domain determined by an invariant will do.

The main objectives in this paper were: first to formulate a general and logically precise theory of property structures based on this generalized notion of structure abstraction; and second, to show that the resulting account explicates the informal talk about structural information in work on invariants. Specifically, the aim here was to show how several operations with invariants typical in mathematical practice can be modeled or explained in our framework. This includes the comparison of invariants in terms of their relative distinctive power as well as the bundling of incomparable invariants.

It was mentioned in the Introduction that the general structuralist *slogan* that mathematics investigates structures is clearly in need of further qualification. What type of structures are there in mathematics? How can we make sense of them philosophically? One possible answer to these questions was given here: sometimes mathematicians study the structural information about objects expressed by invariants. A good way to understand this invariant structuralism is in terms of *in re* property structures. There are certainly other types of structures in mathematics that will call for a different philosophical explication. Moreover, as we saw, a natural alternative framework for the treatment of such structures is category theory. Spelling them out in detail is work for another day.

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